

Differentiability and Differentials.

A function $u = f(x, y)$ is said to be differentiable at a point (x, y) of its domain if the increment δu in the function corresponding to any arbitrary aligned increments δx and δy in x and y respectively, is expressed in the form

$$\begin{aligned} \delta u &= f(x + \delta x, y + \delta y) - f(x, y) \\ &= A\delta x + B\delta y + \delta x\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y) \end{aligned}$$

Where A and B are constants independent of δx and δy and $\phi(\delta x, \delta y)$, $\psi(\delta x, \delta y)$ are functions of $\delta x, \delta y$ such that

$$\lim_{(\delta x, \delta y) \rightarrow (0, 0)} \frac{\phi(\delta x, \delta y)}{\sqrt{\delta x^2 + \delta y^2}} = 0 = \lim_{(\delta x, \delta y) \rightarrow (0, 0)} \frac{\psi(\delta x, \delta y)}{\sqrt{\delta x^2 + \delta y^2}}$$

Here $(A\delta x + B\delta y)$ is the linear part in δx and δy and is called the differential of $f(x, y)$ at (x, y) and it is denoted by du .

$$\text{Thus } du = A\delta x + B\delta y \quad \text{--- (2)}$$

If in the equation (1), we regard y as fixed i.e. $\delta y \rightarrow 0$ and x as variable,

then we have

$$\begin{aligned} \delta u &= A\delta x + \delta x\phi(\delta x, 0) \\ \Rightarrow \frac{\delta u}{\delta x} &= A + \phi(\delta x, 0) \\ \therefore A &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} \right) \text{ since } \phi \rightarrow 0 \end{aligned}$$

$$= \frac{\partial u}{\partial x} \text{ since } y \text{ is regarded as a const.}$$

Similarly, if we regard x as fixed and y as variable, then we get $B = \frac{\partial u}{\partial y}$.

Hence from (2)

$$du = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \quad \text{--- (3)}$$

Hence if $u = f(x, y)$ is differentiable at (x, y) , then the function must possess the partial derivatives f_x and f_y at the point (x, y) .

This is the necessary condition for

differentiability of a function $f(x,y)$ at a given point (x,y) . But its converse is not true, for there are functions which have partial derivatives but are non-differentiable at a point of their domain.

Theorem: If the function $f(x,y)$ is differentiable at a point (a,b) in its domain, then the partial derivatives f_x and f_y both exist and are finite.

Proof: — Since the function is differentiable at (a,b) , therefore
$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$$

Where A and B are constants independent of h and k and $\phi(h,k)$, $\psi(h,k)$ are functions of h and k so that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{\phi(h,k)}{h} = 0 = \lim_{(h,k) \rightarrow (0,0)} \frac{\psi(h,k)}{k}$$

Now, putting $k=0$ and passing to the limit as $h \rightarrow 0$, we find that

$$\lim_{h \rightarrow 0} [f(a+h) - f(a,b)] = A$$

i.e. $f_x(a,b) = A$ Similarly, $f_y(a,b) = B$.