Chapter 1

Basic Concepts of Partial Differential Equations

BY

Prof. D. C. Sanyal

Retired Professor Of Mathematics

University Of Kalyani

West Bengal, India

E-mail : *dcs_klyuniv@yahoo.com*

Module-1: Basic Ideas

1. Introduction

Partial differential equations were originated from the study of surfaces in geometry and a wide variety of problems arising in mechanics. A good number of famous mathematicians investigated numerous problems presented by partial differential equations during the nineteenth century, as these equations can express many fundamental laws arising frequently in science and engineering. In fact, such type of equations have been found to be essential to the theory of surfaces and to the solutions of physical problems, these two areas being linked by the bridge of the calculus of variations. Partial differential equations play a significant role in modern mathematics and other branches of sciences, economics, management, technology and so on.

It is known that almost all physical phenomena obey mathematical laws which can be formulated by differential equations. This striking fact was first noted by Isaac Newton (1642 - 1727) after the formulation of the laws of mechanics. Later, it was found that many partial differential equations govern many physical, chemical and biological phenomena.

From the very beginning, considerable attention has been given to find the solutions of differential equations with a geometric approach. In fact, the families of curves and surfaces can be defined by differential equations which can be studied geometrically. The curves, known as *characteristic curves*, are very useful in finding a surface containing the curve and satisfying a given differential equation.

The study of first-order partial differential equations received the attention to the researchers after the work of A. C. Clairaut (1713 - 1765) on earth's shape. In 1770, Lagrange initiated a systematic study on first-order nonlinear partial differential equations. The author also described the geometrical content of a first-order partial differential equation and developed the method of characteristics in finding the general solutions of quasi-linear equations.

Earlier, the theory of second-order linear partial differential equations was mainly concentrated to mechanics and physics. All these equations can be classified into three types, viz. Laplace (or potential) equation's heat (or diffusion) equation and wave equation. These equations give much information about more general second order linear partial differential equations.

During the second half of nineteenth century, considerable attention has also been given regarding the existence, uniqueness and stability of solutions of partial differential equations. With the advent of new ideas , new methods and applications, both analytical and numerical studies of these equations are in progress. However, there are a large number of problems in partial differential equations which are intractable. These equations are nonlinear and get applications in diversified fields like continuum mechanics, plasma physics, nonlinear optics, biomathematics, quantum field theory and so on.

2. Partial Differential Equations: Definition

A differential equation containing a number of independent variables, one dependent variable along with one or more partial derivatives of the dependent variable, is called a *partial differential equation*. In general, it can be written in the form

$$f(x, y, \cdots, u; u_x, u_y, \cdots, u_{xx}, u_{xy}, \cdots) = 0, \tag{1}$$

where the equation is considered in a domain Ω of the n-dimensional space R^n in the independent variables $x, y, \cdots; u$ is an unknown function of these variables and $u_x, u_y, \cdots, u_{xx}, u_{xy}, \cdots$ are the partial derivatives of u, the subscripts on u denoting differentiations, e.g.

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \text{ etc.}$$

The functions $u = u(x, y, \cdots)$ which satisfy the equation (1), are called the solutions of the equation(1) provided that such functions exist. From these many possible solutions, we have to select one by introducing suitable conditions.

For example, the equations

$$\begin{aligned} &u u_{xy} - u_x u_y = 0, \\ &u_x^2 + u_{xx}^2 + u_{xy} - \sin u = e^y \\ &u_{xx} + u_{yy} = 0 \end{aligned} \tag{2}$$

are partial differential equations. The function $u(x, y) = e^x \sin y$ is a solution of the last equation of (2) as can be easily verified.

The highest order of the partial derivative appearing in the partial differential equation is the *order* of the equation. For example, the equation

$$u_{xx} + 2yu_{xy} + 3xu_y = \cos x, \text{ or } uu_{xy} - u_x u_y = 0 \tag{3}$$

is a second-order partial differential equation and

$$u_{xxx} + u_{xyy} + xu_{yy} + \log u = 0 \tag{4}$$

is a third-order partial differential equation.

A partial differential equation is said to be *linear*, if it is linear in the unknown function and all its derivatives, whose coefficients depend only on the independent variables. For example, the equation

$$u_{xx} + 2yu_{xy} + 3xu_{yy} = 4\sin x \tag{5}$$

is linear.

A partial differential equation is said to be *quasi-linear*, if it is linear in the highest-ordered derivative of the unknown function. For example, the equation, known as Korteweg de Vries (or KdV equation), given by

$$u_t + cuu_x + u_{xxx} = 0 \tag{6}$$

is a third-order quasi-linear partial differential equation.

A partial differential equation is called *semi-linear*, if it is linear in the highest-ordered derivative of the unknown function and a function containing the dependent and/or independent variables. For example, the Klein-Gordon equation

$$u_{tt} - c^2\nabla^2 u + m^2 u + \gamma u^p = 0, \ (p \geq 2 \text{ is an integer}) \tag{7}$$

or the conformal scalar curvature equation

$$\nabla^2 u + K(x)e^{2u} = 0, \tag{8}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \cdots$, is semi linear. Equation of the form (7) arises in quantum field theory (γ being coupling constant) while equation of the type (8) occurs in differential geometry to study the scalar curvature of Riemann matrices which are Euclidean; the metric $(dx^2 + dy^2)e^{2u}$ has a Gauss curvature $K(x,y)$ if u satisfies the equation (8).

Equations which do not belong to above types are known as *nonlinear* partial differential equations. For example, the Monge-Ampére equation

$$\det(u_{x_i x_j}) = f(x,u) \tag{9}$$

arising in differential geometry occurs in a nonlinear way.

The most general second-order linear partial differential in n independent variables $x_1,\ x_2,\cdots,\ x_n$ has the form

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + f u = g, \tag{10}$$

where, without loss of generality, we assume $a_{ij} = a_{ji}$ and that each of a_{ij}, b_i, f and g is a function of n independent variables. If $g = 0$, the equation (10) is said to be *homogeneous*; otherwise, it is a *non-homogeneous* partial differential equation.

The general solution of a linear partial differential equation depends on arbitrary functions in contrast to a linear ordinary differential equation in which an nth order differential equation is a family of functions depending on n arbitrary constants. For instance, consider the equation with $u = u(x,y)$:

$$u_{xy} = 0$$

which, on integration with respect to x leads to

$$u_y(x,y) = g(y).$$

Integrating again with respect to y, we get

$$u(x,y) = h(y) + f(x),$$

where $f(x)$ and $h(y)$ are arbitrary functions.

If we suppose that $u = u(x,y,z)$ and

$$u_{xx} = c$$

where c is constant, then its general solution is

$$u(x,y,z) = x^2 + xf(y,z) + g(y,z)$$

in which f and g are arbitrary functions of y and z.

Now, for linear homogeneous ordinary differential equation of order n, a linear combination of n linearly independent solution is a solution. But, this is not so for partial differential equation since the solution space of every homogenous linear partial differential equation is infinite dimensional. For example, the differential equation

$$u_x + u_y = 0 \tag{11}$$

where $u = u(x,y)$, on substitution $\xi = x+y$, $\eta = x-y$, can be transformed into the equation

$$u_\xi = 0$$

whose general solution is

$$u(x,y) = f(x-y)$$

where $f(x-y)$ is an arbitrary function and so each of the functions $(x-y)^n$, $\sin(x-y)$, $\exp\{n(x-y)\}, \cdots$, where $n = 1, 2, \cdots$, is a solution of the equation (11).

3. Mathematical Problems

The problem of determining an unknown function of a partial differential equation is solved by satisfying supplementary conditions like initial and/or boundary conditions. As an illustration, consider a thin homogeneous perfectly flexible string under uniform tension along the x-axis in its equilibrium position, the ends of the string being fixed at $x = 0$ and $x = L$. The string is pulled aside a short distance and then released. In the absence of external forces which corresponds to free vibrations, the subsequent motion of the string is described by the solution $u(x,\ t)$ of the following problem:

$$\text{Partial differential equation: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L, \quad t > 0$$

$$\text{Initial conditions: } u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x)$$

$$\text{Boundary conditions: } u(0,t) = 0, \ u(L,t) = 0, \quad t > 0$$

This problem is known as *initial-boundary value problem.*

If we like to consider a problem of unbounded medium, then the solution can be obtained uniquely by prescribing initial conditions only. The corresponding problem is known as *initial-value problem* or *Cauchy problem*. For example, in investigating the one-dimensional heat flow for an infinite region subject to an initial temperature $f(x)$, the problem is described by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$
$$\text{and} \quad u(x,0) = f(x), \quad -\infty < x < \infty.$$

A mathematical problem is said to be *well-posed* if it satisfies the following requirements:

(i) *existence*, i.e. there is at least one solution,

(ii) *uniqueness*, i.e. there is at most one solution,

(iii) *continuity*, i.e. the solution depends continuously on the data.

Example 1: Find the general solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = 0$.

Solution: Putting $\frac{\partial u}{\partial x} = v$, the given equation reduces to $\frac{\partial v}{\partial x} + v = 0$, which gives an integration w.r.t. x, $\frac{\partial u}{\partial x} = v = a(y)e^{-x}$ and a further integration leads to the general solution as

$$u(x,y) = b(y) - a(y)e^{-x},$$

$a(y)$ and $b(y)$ being arbitrary functions of y.

Example 2: Show that the general solution of $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ is given by $u(x,t) = f(x - ct) + g(x + ct)$, where f and g are arbitrary twice differentiable functions.

Solution:

$$\begin{aligned} \text{Since} \quad u(x,t) &= f(x-ct) + g(x+ct), \quad \text{so} \\ \frac{\partial^2 u}{\partial t^2} &= c^2 f''(x-ct) + c^2 g''(x+ct) \\ \text{and} \quad \frac{\partial^2 u}{\partial x^2} &= f''(x-ct) + g''(x+ct) \end{aligned}$$

It then readily follows that $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ and thus $u(x,t) = f(x-ct) + g(x+ct)$ is the general solution of this equation.

Example 3: Show that $u = f(xy)$, where f is an arbitrary differential function, satisfies the equation

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

and hence verify that $\cos(xy)$, $\log(xy)$, e^{xy} are also solutions of this equation.

Solution: We have $u = f(xy)$, so that $\frac{\partial u}{\partial x} = yf'(xy)$, $\frac{\partial u}{\partial y} = xf'(xy)$ and thus $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xyf'(xy) - yxf'(xy) = 0$. Hence $u = f(xy)$ satisfies the equation $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$.

Now, if $u = \cos(xy)$, then $\frac{\partial u}{\partial x} = -y\sin(xy)$, $\frac{\partial u}{\partial y} = -x\sin(xy)$ so that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = -xy\sin(xy) + yx\sin(xy) = 0$. Thus $\cos(xy)$ is a solution of the given equation. Similarly, $\log(xy)$ and e^{xy} are also solutions of the equation.

Example 4: If u satisfies the Laplace's equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, then show that xu and yu satisfy the biharmonic equation $\nabla^4 \begin{pmatrix} xu \\ yu \end{pmatrix} = 0$, but they do not satisfy Laplace's equation.

Solution:

$$\begin{aligned}
\text{We have } \nabla^2(xu) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(xu) = 2\frac{\partial u}{\partial x} + x\nabla^2 u = 2\frac{\partial u}{\partial x} \neq 0 \\
\text{and } \nabla^4(xu) &= \nabla^2\left(2\frac{\partial u}{\partial x}\right) = 2\frac{\partial}{\partial x}\left(\nabla^2 u\right) = 0, \\
\text{Similarly, } \nabla^2(yu) &= 2\frac{\partial u}{\partial y} \neq 0, \quad \nabla^4(yu) = 0.
\end{aligned}$$

Hence the result.

4. Surfaces and Curves

Let (x, y, z) be the coordinates of a point P referred to a three-dimensional cartesian system of axes and the coordinates are connected by a relation of the form

$$F(x, y, z) = 0 \tag{12}$$

This equation represents the equation of a surface on which the point P lies. To verify this, we first note that the increments $(\delta x, \ \delta y, \ \delta z)$ in (x, y, z) are related by the relation

$$\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z = 0$$

in which any two can be chosen arbitrarily. Thus, in every neighbourhood of the point P, there exists points $P'(x+\xi, y+\eta, z+\zeta)$ satisfying the relation (12) in which any two of the variables, ξ, η, ζ can be chosen arbitrarily and third is given by

$$\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial z} = 0.$$

The projection of the initial direction $\overline{PP'}$ on the xy-plane can be chosen in arbitrary way. Hence the equation (12) represents a relation satisfied by points lying on the surface.

Next, consider a set of relations of the type

$$x = f_1(u,\ v),\ y = f_2(u,\ v),\ z = f_3(u,\ v) \tag{13}$$

such that there corresponds a set of values to each pair of u, v and hence a point in space. However, it is to be noted that every point in space does not correspond to a pair of values of u and v. The solutions of the first two equations of (13) express u and v as functions of x and y of the form

$$u = F_1(x,y), \ \ v = F_2(x,y)$$

which, when substituted in the third equation of (13) determines z as function of x and y of the type

$$z = f(x,y).$$

Thus the functional relation (12) between the coordinates shows that the point (x,y,z) lies on a surface and that any point determined from them always lies on a fixed surface.

Equations (13) represent *parametric equations* of the surface. It is to be noted that parametric equations of a surface are not unique. For example, the two sets of parametric equations

$$\begin{aligned} x &= a\sin u\cos v,\ \ y = b\sin u\sin v,\ \ z = c\cos u \\ \text{and}\ \ x &= \frac{2av}{1+v^2}\sin u,\ \ y = \frac{2bv}{1+v^2}\cos u,\ \ z = c\frac{1-v^2}{1+v^2} \end{aligned}$$

lead to the same ellipsoidal surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

A surface may be thought to be generated by a curve. Suppose, a point with coordinates (x,y,z) satisfying the equation (12) lies on a plane $z = k$, where k is constant. Then

$$F(x,y,k) = 0, \ \ z = k \tag{14}$$

which shows that the point (x,y,z) lies on a curve Γ in the plane $z = k$. For example, if we consider an elliptic paraboloid $E: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$, where c is constant, then the section of this elliptic paraboloid by the plane $z = k$ is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}, \; z = k \tag{15}$$

which represents an ellipse with centre at the point $(0,0,k)$ if k is positive. Thus the whole surface of E is generated by ellipses like (15) with centres on the z-axis as k varies.

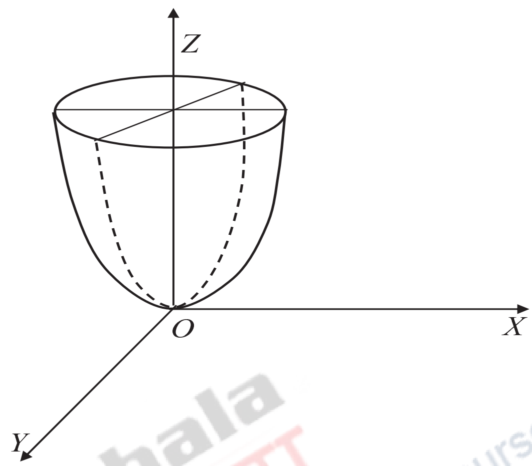

Fig. 1

Alternatively, we can say that the curve represented by equations (14) is obtained by the intersection of the surface (12) with a plane $z = k$, or more generally, the curve may be thought of the intersection of two surfaces. For, if a point (x,y,z) lies on both the surfaces $S_1: F_1(x,y,z) = 0$ and $S_2: F_2(x,y,z) = 0$, then the points common to both S_1 and S_2 must satisfy the equations

$$F_1(x,y,z) = 0, \quad F_2(x,y,z) = 0$$

so that the locus of a point whose coordinates satisfy these equations is a curve in space.

We may represent a curve by parametric equations of the form

$$x = f_1(t), \;\; y = f_2(t), \;\; z = f_3(t), \tag{16}$$

t being a continuous variable. For, if a point P has coordinates (x,y,z), then by eliminating t between the equations $x = f_1(t)$, $y = f_2(t)$ and $x = f_1(t)$, $z = f_3(t)$, we find that the point P lies on a curve whose equations are

$$\phi_1(x,y) = 0, \quad \phi_2(x,z) = 0.$$

Now, consider a point P on the curve, taken to be arbitrary,

$$x = x(s), \;\; y = y(s), \;\; z = z(s), \tag{17}$$

s being the arc length parameter, measured from some fixed point P_0 along the curve and $Q\,(x(s+\delta s), y(s+\delta s), z(s+\delta s))$ be another point on it such that arc$PQ = \delta s$ and chord $PQ = \delta L$.

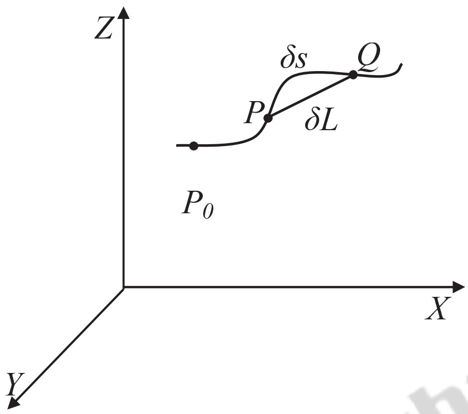

Fig. 2

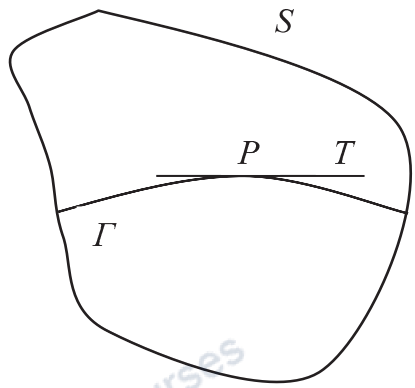

Fig. 3

Then $\lim\limits_{\delta s \to 0} \frac{\delta L}{\delta s} = 1$. Now the direction cosines of the chord PQ are

$$\begin{aligned}
& \frac{x(s+\delta s) - x(s)}{\delta L}, \;\; \frac{y(s+\delta s) - y(s)}{\delta L}, \;\; \frac{z(s+\delta s) - z(s)}{\delta L} \\
\text{i.e.} \quad & \frac{\delta s}{\delta L}\left\{\frac{dx}{ds} + o(\delta s), \;\; \frac{dy}{ds} + o(\delta s), \;\; \frac{dz}{ds} + o(\delta s)\right\} \;\; \text{(by Mclaurin's theorem)} \\
\Rightarrow \quad & \frac{dx}{ds}, \;\frac{dy}{ds}, \;\frac{dz}{ds} \quad \text{as} \quad \delta s \to 0, \quad \text{i.e.} \;\; Q \to P
\end{aligned}$$

and so the chord PQ takes the direction of the tangent to the curve at P. Hence, the direction cosines of the tangent to the curve at the point $P(x, y, z)$ are $(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds})$. Thus, the arbitrariness of the curve (17) is true.

Next, let us suppose that the curve Γ given by the equations (17) lies on the surface $S : F(x, y, z) = 0$ so that the point $(x(s), y(s), z(s))$ lies on it and hence

$$F\,(x(s), y(s), z(s)) = 0. \tag{18}$$

If the curve Γ lies entirely on the surface S, then the equation (18) becomes an identity for all values of s. Differentiating (18) with respect to s, we get

$$\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0 \tag{19}$$

Thus the tangent to the curve Γ at the point P is perpendicular to the line of direction ratios $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$. The curve Γ is arbitrary except that it passes through the point P and lies entirely on the surface S. Also, since the line with direction ratios $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ is perpendicular to the tangent to every curve lying on S and passes through P, so this must be normal to the surface S at P.

Now, suppose the surface S has the equation $z = f(x,y)$ so that

$$F(x,y,z) = f(x,y) - z = 0. \tag{20}$$

Then the direction cosines of the normal at any point (x,y,z) of the surface are

$$\frac{1}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \frac{1}{\sqrt{p^2+q^2+1}}(p, q, -1), \tag{21}$$

where $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ and since $\frac{\partial F}{\partial x} = p$, $\frac{\partial F}{\partial x} = q$, $\frac{\partial F}{\partial z} = -1$.

The equation of the tangent plane Π at the point (x,y,z) to the surface $S : F(x,y,z) = 0$ is

$$(X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0 \tag{22}$$

(X,Y,Z) being the coordinates of another point on Π. Similarly, the tangent plane Π, at the point (x,y,z) to the surface $S_1 : G(x,y,z) = 0$ has the equation

$$(X-x)\frac{\partial G}{\partial x} + (Y-y)\frac{\partial G}{\partial y} + (Z-z)\frac{\partial G}{\partial z} = 0 \tag{23}$$

The intersection of the planes Π and Π_1 is the line L tangent at P to the curve Γ generated by the surfaces S and S_1, the equation of the line being

$$\frac{(X-x)}{\frac{\partial(F,G)}{\partial(y,z)}} = \frac{(Y-y)}{\frac{\partial(F,G)}{\partial(z,x)}} = \frac{(Z-z)}{\frac{\partial(F,G)}{\partial(x,y)}} \tag{24}$$

Thus the line has direction cosines $\left\{\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)}\right\}$.

Example 5: Show that the condition that the surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$ should touch is that the eliminant of x,y,z from these equations and the equations $F_x : G_x = F_y : G_y = F_z : G_z$ should hold.

Hence find the condition that the plane $lx+my+nz+p = 0$ should touch the central conicoid $ax^2+by^2+cz^2 = 1$

Solution: First part.

If the given surfaces touch each other at some point (x,y,z), then they must have the common tangent at this point and so the equations

$$(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0 \text{ and } (X-x)\frac{\partial G}{\partial x}+(Y-y)\frac{\partial G}{\partial y}+(Z-z)\frac{\partial G}{\partial z}=0$$

must be identical and, therefore, $F_x : G_x = F_y : G_y = F_z : G_z$.

Thus, the required equation is obtained by eliminating x,y,z from the given equations of surfaces and the equations $F_x : G_x = F_y : G_y = F_z : G_z$.

Second part.

Let $F(x,y,z) = lx+my+nz+p = 0$ and $G(x,y,z) = ax^2+by^2+cz^2-1 = 0$. These two surfaces touch each other at the point if the condition $F_x : G_x = F_y : G_y = F_z : G_z$ does hold, i.e. if

$$\frac{l}{ax}=\frac{m}{by}=\frac{n}{cz}=\frac{1}{k}, \text{ (say), where } k \text{ is constant}$$

Thus $x = kl/a, y = km/b, z = kn/c$ and then the equation $F(x,y,z) = 0$ gives $k = -\frac{1}{p}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)$.

Lastly, from the equation $G(x,y,z) = 0$, we get

$$k^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)=1 \Rightarrow p^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$$

This is the required condition.