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B.SC III (H)

Paper - V

Unit - Gp A

Topic - Orthogonal Curvilinear coordinates

Expt. No. Date

B.SC III (Honours)

Gp-A Methods of mathematical Physics

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Chapter 1 Co-ordinate systems

Orthogonal Curvilinear coordinates

In Cartesian coordinates the position of a point $P(x, y, z)$ is determined by the intersection of three mutually perpendicular planes x = constant,

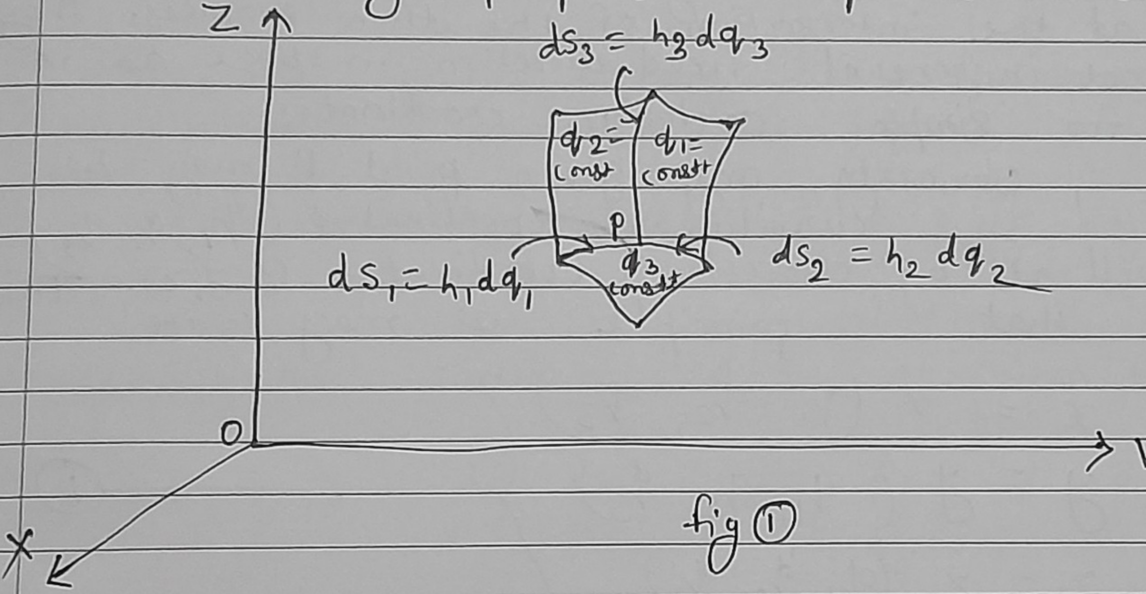

fig ①

y = constant and z = constant. Imagine that we superimpose on this system three other families of surfaces. The surfaces of any family need not be parallel and they need not be planes. The three new families of surfaces need not be mutually perpendicular. Let the three new families of surfaces described by q_1 = constant, q_2 = constant, q_3 = constant intersect at point P. The values of q_1, q_2, q_3 for the three surfaces intersecting at P are called the curvilinear

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coordinates of P. The three new surfaces are then called the coordinate surfaces or curvilinear surfaces.

If the coordinate surfaces are mutually perpendicular at every point $P(x, y, z)$ then the curvilinear coordinates (q_1, q_2, q_3) are said to be orthogonal curvilinear coordinates. The coordinate surfaces intersect pairwise in three curves called the coordinate lines (or coordinate curves). The coordinate axes are determined by the tangents to the coordinate lines at the intersection of the three surfaces. They are not in general fixed direction in space, as is true for simple cartesian coordinates.

Obviously any given point P may be identified by curvilinear coordinates (q_1, q_2, q_3) as well as by cartesian coordinates (x, y, z). This means that in principle we may write

$$\left.\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned}\right\} \quad \text{——— (1)}$$

with inverses

$$\left.\begin{aligned} q_1 &= q_1(x, y, z) \\ q_2 &= q_2(x, y, z) \\ q_3 &= q_3(x, y, z) \end{aligned}\right\} \quad \text{——— (2)}$$

with each family of surface q_i = constant we can associate a unit vector $\hat{u}_i$ normal to each surface q_i = constant and in the direction of increasing q_i.

The partial differentiation of eqn (1) yields

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$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$dy = \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2 + \frac{\partial y}{\partial q_3} dq_3$$

$$dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \frac{\partial z}{\partial q_3} dq_3 \quad \text{— (3)}$$

Hence the square of the distance between two neighbouring points is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{ij} h_{ij}^2 \, dq_i \, dq_j \quad (i,j = 1,2,3) \quad \text{— (4)}$$

where the coefficients h_{ij}^2 are given by

$$h_{ij}^2 = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} \quad \text{— (5)}$$

The most useful coordinate systems are orthogonal ones, i.e. the systems in which surfaces always intersect at right angles. At this point we limit ourselves to orthogonal coordinate systems which means

$$h_{ij} = 0, \quad i \neq j \quad \text{— (6)}$$

Now to simplify the notation, we write $h_{ii} = h_i$ so that equation (4) yields

The specific coordinate systems are described by specifying the scale factors h_1, h_2 and h_3.

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The distance between any two points on a coordinate line is called the line element. When the variation is limited to any given q_i holding the other q's constant, then line element, from equⁿ (4), given by

$$ds_i = h_i \, dq_i \qquad \text{——— (8)}$$

from this equation it may be noted that the three curvilinear coordinates q_1, q_2, q_3 need not be lengths. The scale factors h_i may depend on q's and they may have dimensions. The product $h_i \, dq_i$ must have the dimensions of length.

from equⁿ (8) we may immediately develop the surface and volume elements.

The three possible surface elements in orthogonal systems thus become

$$dS_{ij} = ds_i \, ds_j = h_i h_j \, dq_i \, dq_j \quad (i, j = 1, 2, 3; \; i \neq j) \qquad \text{——— (9)}$$

and the volume element.

$$d\tau = ds_1 \, ds_2 \, ds_3 \; \cancel{= h_1 h_2 \, dq_1 \, dq_2 \, dq_3 \; (i, j = 1, 2, 3; \; i \neq j)}$$

$$= h_1 h_2 h_3 \, dq_1 \, dq_2 \, dq_3 \qquad \text{——— (10)}$$