Theorem :— If at the point (x,y), the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ of the function $u = f(x,y)$ exist and are continuous, then the function is differentiable at that point.

Proof:— The condition that the functions $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous implies that these functions exist in a certain nbd. $(x-\delta, x+\delta ; y-\delta, y+\delta)$ of the point (x,y).

Let $(x+h, y+k)$ be any point of this nbd.

Then
$$f(x+h, y+k) - f(x,y) = \{f(x+h, y+k) - f(x, y+k)\} + \{f(x, y+k) - f(x,y)\} \quad \text{---(1)}$$

The R.H.S. of (1) consists of the sum of two differences. Consider the first difference $\{f(x+h, y+k) - f(x, y+k)\}$. Here the second argument of both the terms, namely $y+k$ are the same.

Therefore we can regard the function $f(x, y+k)$ of one variable x derivable at every point of the interval $(x, x+h)$ with respect to x derivable at every point of the interval $(x, x+h)$ in the nbd. of the point (x,y).

Hence applying the mean-value theorem, we have

$$f(x+h, y+k) - f(x, y+k) = h f_x(x+\theta_1 h, y+k)$$

Where $0 < \theta_1 < 1$. θ_1 being a function of h and k. (2)

Since $f_x(x,y)$ is continuous at (x,y), therefore we can write

$$f_x(x+\theta_1 h, y+k) - f_x(x,y) = \phi(h,k) \quad \text{---(3)}$$

Where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Thus from (2) and (3)

$$f(x+h, y+k) - f(x, y+k) = h[f_x(x,y) + \phi(h,k)] \quad \text{---(4)}$$

Similarly for the second difference on the R.H.S. of (1), we have

$$f(x, y+k) - f(x,y) = k f_y(x, y+\theta_2 k) \quad \text{---(5)}$$

Where $0 < \theta_2 < 1$: θ_2, being a function of k.
Since $f_x(x,y)$ is continuous at (x,y), therefore
We can write

$$f_y(x, y+\theta_2 k) - f_y(x,y) = \psi(k)$$

Where $\psi(k) \to 0$ as $k \to 0$

Hence from (5)

$$f(x, y+k) - f(x,y) = k\left[f_x(x,y) + \psi(k)\right] \quad \text{——— (6)}$$

Thus from (1), (4) and (6)

$$f(x+h, y+k) - f(x,y) = h\left[f_x(x,y) + \phi(h,k)\right] + k\left[f_y(x,y) + \psi(k)\right]$$

$$= h\, f_x(x,y) + k\, f_y(x,y) + h\,\phi(h,k) + k\,\psi(k)$$

Where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$ and
$\psi(k) \to 0$ as $k \to 0$

Hence ~~f(x)~~ $f(x,y)$ is differentiable at (x,y)