Differentiability and Differentials.

A function $u = f(x,y)$ is said to be differentiable at a point (x,y) of its domain if the increment δu in the function corresponding to any arbitrary assigned increments δx and δy in x and y respectively, is expressed in the form

$$\delta u = f(x+\delta x, y+\delta y) - f(x,y)$$
$$= A\delta x + B\delta y + \delta x\,\phi(\delta x,\delta y) + \delta y\,\psi(\delta x,\delta y) \quad (1)$$

Where A and B are constants independent of δx and δy and $\phi(\delta x,\delta y)$, $\psi(\delta x,\delta y)$ are functions of $\delta x, \delta y$ such that

$$\underset{(\delta x,\delta y)\to(0,0)}{Lt} \phi(\delta x,\delta y) = 0 = \underset{(\delta x,\delta y)\to(0,0)}{Lt} \psi(\delta x,\delta y).$$

Here $(A\delta x + B\delta y)$ is the linear part in δx and δy and is called the differential of $f(x,y)$ at (x,y) and it is denoted by du.

Thus $\quad du = A\delta x + B\delta y$ ——— (2)

If in the equation (1), we regard y as fixed i.e. $\delta y \to 0$ and x as variable, then we have

$$\delta u = A\delta x + \delta x\,\phi(\delta x, 0)$$
$$\Rightarrow \frac{\delta u}{\delta x} = A + \phi(\delta x, 0)$$
$$\therefore A = \underset{\delta x\to 0}{Lt}\left(\delta u/\delta x\right) \quad \text{since } \phi \to 0$$
$$= \frac{\partial u}{\partial x} \quad \text{since } y \text{ is regarded as a const.}$$

Similarly, if we regard x as fixed and y as variable, then we get $B = \dfrac{\partial u}{\partial y}$.

Hence from (2)

$$du = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \quad \text{———— (3)}$$

Hence if $u = f(x,y)$ is differentiable at (x,y), then the function must posses the partial derivatives f_x and f_y at the point (x,y).

This is the necessary condition for

differentiability of a function $f(x,y)$ at a given point (x,y). But its converse is not true, for there are functions which does have partial derivatives but are non-differentiable at a point of their domain.

Theorem: If the function $f(x,y)$ is differentiable at a point (a,b) in its domain, then the partial derivatives f_x and f_y both exist and are finite.

Proof. Since the function differentiable at (a,b), therefore

$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k) \quad \text{---(1)}$$

Where A and B are constants independent of h and k and $\phi(h,k)$, $\psi(h,k)$ are function of h and k so that

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = 0 = \lim_{(h,k)\to(0,0)} \psi(h,k)$$

Now, putting $k=0$ and passing to the limit as $h \to 0$, we find that

$$\lim_{h\to 0} [f(a+h) - f(a,b)] = A$$

i.e. $f_x(a,b) = A$ Similarly, $f_y(a,b) = B$.