B.sc Part III (H)
Paper - V.

Continuity and Differentiability of a function of two variables.

Continuity : (1) A function $u = f(x,y)$ is said to be continuous at a given point (a,b) of its domain of definition if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Thus $f(x,y)$ is continuous at (a,b) if $|f(x,y) - f(a,b)| < \epsilon$ for all (x,y) satisfying $|x-a| < \delta$, $|y-b| < \delta$.

That is, $f(x,y)$ is continuous at (a,b) if the double limit $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$ exists and is equal to its functional value $f(a,b)$ at (a,b).

(2) Definition — A function $u = f(x,y)$ is said to be continuous at a given point (a,b) if

$$\lim_{\rho\to 0} f(a+h, b+k) = f(a,b) \text{ Where } \rho = \sqrt{h^2+k^2}$$

Thus for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(a+h, b+k) - f(a,b)| < \epsilon \text{ for } \rho < \delta.$$

Theorem: — To prove that the continuity of $f(x,y)$ at (a,b) implies the continuity of $f(x,y)$ as a function of x and as a function of x and as a function of y at a and b respectively but not conversely.

Proof: — Since the function $f(x,y)$ is continuous at (a,b), therefore

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

i.e. Corresponding to an arbitrary small +ve ϵ, we can find a $\delta > 0$

Such that

$$|f(x,y) - f(a,b)| < \epsilon/2 \text{ When } |x-a| < \delta \text{ and } |y-b| < \delta.$$

Now, if (x_1, b), (x_2, b) be any two points such that the range of x is an interval $(a-\delta, a+\delta)$ lying within the domain of definition, then

$$|f(x_1,b) - f(x_2,b)| = |\{f(x_1,b) - f(a,b)\} + \{f(a,b) - f(x_2,b)\}|$$
$$\leq |f(x_1,b) - f(a,b)| + |f(x_2,b) - f(a,b)|$$

$< \epsilon/2 + \epsilon/2 = \epsilon$

i.e. $|f(x,b) - f(a,b)| < \epsilon$ where $|x-a| < \delta$.

Thus $f(x,b)$ is a continuous function of one variable for $x = a$

Similarly $|f(a,y_1) - f(a,y_2)| < \epsilon$

or $|f(a,y) - f(a,b)| < \epsilon$ where $|y-b| < \delta$

ie. $f(a,y)$ is also a continuous function of one variable for $a = a$

Similarly $|f(a,y_1) - f(a,y_2)| < \epsilon$

or $|f(a,y) - f(a,b)| < \epsilon$ where $|y-b| < \delta$.

ie. $f(a,y)$ is also a continuous function of one variable y for $y = b$.