

Group: - Let G be a non-empty set and \circ is binary operation then the system (G, \circ) is called group iff the following postulates are satisfied:

(i) closure law: - For all $a, b \in G$ such that

i.e. $a \circ b \in G$.
i.e. G is closed under the binary operation \circ .

(ii) Associative law: - For all $a, b, c \in G$ such that

$$(a \circ b) \circ c = a \circ (b \circ c)$$

i.e. associative law for the operation \circ

holds.

(iii) Existence of ^{inverse} ~~identity~~ elements: - For every $a \in G$

there exists an element $b \in G$ such that $a \circ b = b \circ a = e$,
 e being the identity element

i.e. the inverse elements of all the elements of G exist. The inverse of a denoted by a^{-1} .

(iv) Existence of identity element: - For all $a \in G$ and

there exists an element $e \in G$ such that $a \circ e = e \circ a = a$.

i.e. the identity element exist in G .

Theorem: - Show that the identity element in a group is unique.

or

Show that a group has a unique neutral element.

Proof: - let suppose that a group (G, \circ) has two identity elements e and e' , where $e \neq e'$.

When e is identity element then from the law of existence of identity element of a group, we have

$$a \circ e = e \circ a = a \quad \text{--- (1) } \forall a \in G$$

And when e' is identity of G then

$$a \circ e' = e' \circ a = a \quad \text{--- (2) } \forall a \in G$$

Here the equality of (1) and (2) show that $e = e'$

This lead our contradiction that $e \neq e'$. Hence our

substitution is wrong. So the identity element of a group is unique.

2. Theorem: - Show that every element of a group has a unique inverse. or

Prove that the inverse element of an element in a group is unique.

Proof: - Let (G, \circ) be a group and a be any element of G .

Suppose x and y are two inverses of a then we have

$$a \circ x = x \circ a = e \quad [\because x \text{ is inverse of } a] \quad \text{--- (1)}$$

$$a \circ y = y \circ a = e \quad [\because y \text{ is inverse of } a] \quad \text{--- (2)}$$

From above two relations, we have

$$a \circ x = a \circ y = e$$

$$\Rightarrow x \circ (a \circ x) = x \circ (a \circ y) \quad [\text{multiply left by } x]$$

$$\Rightarrow [x \circ a] \circ x = (x \circ a) \circ y \quad [\text{By associative law}]$$

$$\Rightarrow e \circ x = e \circ y \quad (\text{by (1)})$$

$$\Rightarrow x = y \quad [\because e \text{ is identity }]$$

This shows that any two inverse of a must be identical. Hence every element of a group G is unique.

Proved