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#3 Ladder Operators L_+ & L_-:

Let $L_+ = L_x + iL_y$ & $L_- = L_x - iL_y$

$[L_z, L_\pm] = ?$

$$[L_z, L_+] = [L_z, L_x + iL_y]$$
$$= [L_z, L_x] + i[L_z, L_y]$$
$$= i\hbar L_y + i(-i\hbar L_x) = i\hbar L_y + \hbar L_x$$
$$= \hbar[L_x + iL_y] = \hbar L_+$$

& $[L_z, L_-] = [L_z, L_x - iL_y]$
$$= [L_z, L_x] - i[L_z, L_y]$$
$$= i\hbar L_y - i(-i\hbar L_x)$$
$$= i\hbar L_y - \hbar L_x = -\hbar(L_x - iL_y)$$
$$= -\hbar L_-$$

Next $[L_+, L_-] = [(L_x + iL_y), (L_x - iL_y)]$
$$= [L_x, L_x] - i[L_x, L_y] + i[L_y, L_x] + [L_y, L_y]$$
$$= 0 - i(i\hbar L_z) + i(-i\hbar L_z) + 0$$
$$= \hbar L_z + \hbar L_z = 2\hbar L_z$$

Next: $[L_x, x] = [L_x x - xL_x]$

$$= \left\{y\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - z\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\right\}$$
$$-x\left\{y\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - z\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\right\}$$

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Now operate both sides on wave fn $\psi(x)$.

$$(L_x x - x L_x)\psi = \left\{ \left(y \frac{\hbar}{i}\frac{\partial}{\partial z}(x\psi) \right) - z \left\{ \frac{\hbar}{i}\frac{\partial}{\partial y}(x\psi) \right\} \right.$$
$$\left. - x \left\{ y \left(\frac{\hbar}{i}\frac{\partial \psi}{\partial z} \right) - z \left(\frac{\hbar}{i}\frac{\partial \psi}{\partial y} \right) \right\} \right]$$
$$= 0$$

or $[L_x, x] = 0$

Sly;
$[L_x, y] = i\hbar z$; $[L_x, z] = -i\hbar y$;
$[L_y, x] = -i\hbar z$; $[L_y, y] = 0$; $[L_y, z] = i\hbar x$;
$[L_z, x] = i\hbar y$; $[L_z, y] = -i\hbar x$; $[L_z, z] = 0$

Sly Relⁿ betⁿ $\vec{L}$ & $\vec{p}$ taking into account the cycling order as:
$x \to y$, $y \to z$, $z \to x$ Then;
$[L_x, p_x] = 0$; $[L_x, p_y] = i\hbar p_z$; $[L_x, p_z] = -i\hbar p_y$

(#) Eqⁿ of Motion in Terms of operators.

Hamiltonian Operator:

$$H = \frac{p^2}{2m} + V(x)$$

H has a real eigen value & is a Hermitian operator.

Operators H does not changes with time but H is a f (x, p).

Time-dependence of the states is contained in the wave-function.

Thus $\dot{p} = dp/dt$ has no Quantum interpretation as such.

If $\underline{A}$ is defined such that the mean (expectation value) in any state ψ is time-derivative of the expectation of the operator A.

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \quad \text{--- (1)}$$

Then
$$H^* \psi^* = -i\hbar \frac{\partial \psi^*}{\partial t} \quad \text{--- (2)}$$

Say ψ is a normalised state.

$$\langle A \rangle = \int \psi^* A \psi \, d\tau \quad \text{--- (3)}$$

Taking time-derivative of (3)

$$\frac{d}{dt}\langle A \rangle = \int \frac{\partial \psi^*}{\partial t} A \psi \, d\tau + \int \psi^* A \frac{\partial \psi}{\partial t} d\tau + \int \psi^* \frac{\partial A}{\partial t} \psi \, d\tau \quad \text{--- (4)}$$

Using (1) & (2), we will have

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar} H\psi$$

&
$$\frac{\partial \psi^*}{\partial t} = -\frac{1}{i\hbar} H^*\psi^*$$

Using these eqns. eqn. 4 takes the form

$$\frac{d}{dt}\langle A\rangle = \int\left(-\frac{1}{i\hbar}H^*\psi^* A\psi\, d\tau\right) + \int \psi^* A \frac{1}{i\hbar} H\psi\, d\tau + \int \psi^* \frac{\partial A}{\partial t}\psi\, d\tau$$

$$= \frac{1}{i\hbar}\int\left(\psi^* A H\psi - H^*\psi^* A\psi\right)d\tau + \int \psi^* \frac{\partial A}{\partial t}\psi\, d\tau \longrightarrow 5$$

As H is a Hermitian operator;

$$\int H^*\psi^*(A\psi)\,d\tau = \int \psi^* H (A\psi)\,d\tau \longrightarrow 6$$

Applying 6 in 5 we will have;

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\int \psi^*(AH - HA)\psi\, d\tau + \int \psi^* \frac{\partial A}{\partial t}\psi\, d\tau \longrightarrow 7$$

Eqn. 7 may be written as;

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle (AH - HA)\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$$

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A,H]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle \longrightarrow 8$$

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where $[A, H] = (AH - HA)$ is the commutation relⁿ betⁿ A & H.

If the operator A does not depend explicitly on time then $\frac{\partial A}{\partial t} = 0$ then the above eqⁿ. becomes

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle[A, H]\rangle \rightarrow (9)$$

Eqⁿ. (8) or Eqⁿ (9) is known as eqⁿ of motion in Q.M.

If A & H commute then $[A, H] = 0$ then

$$\frac{d}{dt}\langle A\rangle = 0 \text{ or } \langle A\rangle = \text{const.}$$

Remark: If operator A associated with any (particular) dynamical variable of the system commutes with the Hamiltonian of the system and does not depend upon time explicitly, then that particular dynamical variable is called a constant of motion of the system.