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Teaching: Online Mode

Further evaluate $[x, p_y]$

$$p_y = \frac{\hbar}{i}\frac{\partial}{\partial y}$$

$$[x, p_y]\psi = (x p_y - p_y x)\psi = x\frac{\hbar}{i}\frac{\partial\psi}{\partial y} - \frac{\hbar}{i}\frac{x\partial\psi}{\partial y}$$

$$\left[\because \frac{\partial x}{\partial y} = 0\right]$$

$$= 0$$

$$\therefore [x, p_y] = 0$$

Similarly $[x, y] = xy - yx = 0$

$$[p_x, p_y]\psi = (p_x p_y - p_y p_x)\psi$$

$$= \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\psi - \left(\frac{\hbar}{i}\frac{\partial}{\partial y}\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi$$

$$= -\hbar^2\frac{\partial^2\psi}{\partial x\partial y} + \hbar^2\frac{\partial^2\psi}{\partial y\partial x} = 0 \quad \left[\because \frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial y\partial x}\right]$$

$[p_x, p_y] \ldots$

Teacher's Signature

(#) Commutation Rel. bet. momentum and Hamiltonian

$$H = \frac{p_x^2}{2m} + V(x)$$

$p_x \rightarrow$ Momentum operator

$V(x) \rightarrow$ P.E. operator

$$\therefore [H, p_x] = \left[\frac{p_x^2}{2m} + V(x), p_x\right]$$

$$= \left[\frac{p_x^2}{2m}, p_x\right] + [V(x), p_x]$$

$$= \frac{1}{2m}[p_x^2, p_x] + [V(x), p_x]$$

$$= 0 + [V(x), p_x]$$

Now operate $[H, p_x]$ on ψ

$$[H, p_x]\psi(x) = [V(x)\, p_x - p_x V(x)]\psi(x)$$

$$= V(x)\frac{\hbar}{i}\frac{\partial \psi(x)}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}\{V(x)\psi(x)\}$$

$$= V(x)\frac{\hbar}{i}\frac{\partial \psi(x)}{\partial x} - \frac{\hbar}{i} V(x)\frac{\partial \psi(x)}{\partial x} - \frac{\hbar}{i}\psi(x)\frac{\partial V(x)}{\partial x}$$

$$= -\frac{\hbar}{i}\frac{\partial V(x)}{\partial x}\psi(x)$$

$$\boxed{\therefore [H, p_x] = i\hbar \frac{\partial V(x)}{\partial x}}$$

When particle is free $[V(x) = 0,$

$[H, p_x] = 0$ for free particle.

If momentum operator in 3D & $V = V(x)$ in 3D then

$$p = \frac{\hbar}{i}\nabla \Rightarrow \boxed{[H, p] = i\hbar\nabla V}$$

(III) The Commutation Rules for the Components of Orbital Angular Momentum.

Say momentum $\to p$ & $r \to$ vector relative to origin

then: $\vec{L} = \vec{r} \times \vec{p}$

or, $(\vec{i}L_x + \vec{j}L_y + \vec{k}L_z) = (\vec{i}x + \vec{j}y + \vec{k}z) \times (\vec{i}p_x + \vec{j}p_y + \vec{k}p_z)$

$$= (yp_z - zp_y)\vec{i} + (zp_x - xp_z)\vec{j} + (xp_y - yp_x)\vec{k}$$

After comparing both sides we will have,

$L_x = yp_z - zp_y = y\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - z\left(\frac{\hbar}{i}\partial/\partial y\right)$

ly $L_y = zp_x - xp_z = z\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) - x\left(\frac{\hbar}{i}\partial/\partial z\right)$

& $L_z = xp_y - yp_x = x\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - y\left(\frac{\hbar}{i}\partial/\partial x\right)$

Commutation Rules:

$$L_xL_y = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= -\hbar^2\left\{y\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial z}\left(x\frac{\partial}{\partial z}\right) - z\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}\right) + z\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial z}\right)\right\}$$

$$= -\hbar^2\left\{y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x} - yx\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial y\partial x} + zx\frac{\partial^2}{\partial y\partial z}\right\}$$

Sly

$$L_y L_x = -\hbar^2 \left\{ zy \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial x \partial y} - xy \frac{\partial^2}{\partial z^2} + x \frac{\partial}{\partial y} + xz \frac{\partial^2}{\partial z \partial y} \right\}$$

So the operation

$L_x L_y - L_y L_x$

$= [L_x, L_x]$

$$= (-\hbar^2) \left\{ y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right\}$$

$\because \frac{\partial^2}{\partial x \partial z} = \frac{\partial^2}{\partial z \partial x}$ & so

when x, y, z are perfect differentials.

$$\text{Thus } [L_x, L_y] = \frac{\hbar}{i} \left\{ \frac{\hbar}{i} \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \right\}$$

$$= -\frac{\hbar}{i} \left\{ \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \right\}$$

$$= -\frac{\hbar}{i} L_z = i\hbar L_z$$

or $\boxed{[L_x, L_y] = i\hbar L_z}$

Sly $\boxed{\begin{aligned} [L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y \end{aligned}}$

Next $[L^2, L_x; L_y; L_z] \to$ Respectively

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x]$$

$$= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$$

But $[L_x^2, L_x] = [L_x L_x, L_x]$

$$= L_x[L_x, L_x] + [L_x, L_x]L_x$$

$$= 0$$

Next; $[L^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$

$$\therefore = [L_y L_y, L_x] + [L_z L_z, L_x]$$

$$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$$

But $[L_x, L_y] = i\hbar L_z \Rightarrow [L_y, L_x] = -i\hbar L_z$

Also; $[L_z, L_x] = i\hbar L_y$ & $[L_x, L_z] = -i\hbar L_y$

Therefore; $[L^2, L_x] = L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z$

$$= 0$$

$$\therefore [L^2, L_x] = 0$$

& Also $[L^2, L_y] = 0$

$$[L^2, L_z] = 0$$

Hence L^2 commutes with all the three compts of Ang. Mom. operator & have simultaneous eigen fns. and are simultaneously measurable.