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Taylor's Series

Theorem :- If $f(z)$ is analytic at all points inside a circular domain D with its centre at $z = z_0$ and radius r_0, then for every z inside D,

$$f(z) = f(z_0) + \frac{f'(z)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

This series is called the Taylor's series of $f(z)$ with centre at z_0.

In particular case when $z_0 = 0$, this series is called the Maclaurin Series of $f(z)$.

Proof:- Let C_0 denote the boundary of the circular domain D. Let z be any fixed point inside the circle C_0 and such that $|z - z_0| = r$ where $r < r_0$. Let z' denote any point on a circle $|z' - z_0| = r_1$, denoted by C where $r < r_1 < r_0$. Then, as shown in fig ①, z is inside C and $f(z)$ is analytic within and on C; hence from Cauchy's integral formula it follows that

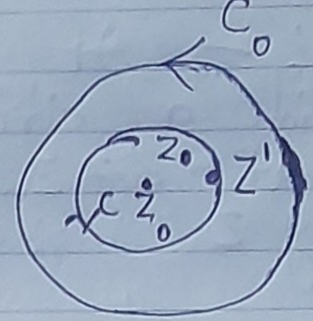

fig ①

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')\,dz'}{z' - z}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z')\,dz'}{(z' - z_0) - (z - z_0)}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z')\,dz'}{(z' - z_0)\left[1 - \frac{z - z_0}{z' - z_0}\right]} \quad \text{①}$$

from geometric progression

$$1 + q + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q}$$

we obtain the relation

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^n + \frac{q^{n+1}}{1-q} \quad \text{—} \ (2)$$

Now for point z interior to C, we have

$|z - z_0| < |z' - z_0|$ since z' is on C.

$$\therefore \frac{|z - z_0|}{|z' - z_0|} < 1$$

By setting $q = \frac{z - z_0}{z' - z_0}$ in (2), we get

$$\frac{1}{1 - \left[\frac{(z-z_0)}{(z'-z_0)}\right]} = 1 + \frac{z - z_0}{z' - z_0} + (z - z_0)^2 + \cdots + \left(\frac{z - z_0}{z' - z_0}\right)^n + \frac{\left[(z-z_0)/(z'-z_0)\right]^{n+1}}{\left[1 - \frac{(z-z_0)}{z'-z_0}\right]} \quad \text{—} \ (3)$$

Substituting this in eqn (1) and taking powers of $(z - z_0)$ out from under the integral sign, since z and z_0 are constant, we get

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{z'-z_0} + \frac{(z-z_0)}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)^2}$$

$$+ \cdots + \frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + R_n(z) \qquad \text{—— (4)}$$

where $R_n(z)$ is given by

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\int \frac{f(z')\,dz'}{(z'-z_0)^{n+1}(z'-z)} \qquad \text{—— (5)}$$

Using the integral Cauchy's formulas viz,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n = 1, 2, 3 \cdots) \qquad \text{—— (6)}$$

expression (4) may be written in the form

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0)$$

$$+ \cdots + \frac{(z-z_0)^n}{n!}f^{(n)}(z_0) + R_n \qquad \text{—— (7)}$$

This expression is called Taylor's formula with remainder R_n.

Since $f(z')$ is analytic, it has derivatives of all orders; hence we may take n in eqn (7) as large as we please. If we let n approach infinity, we obtain from (7) the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)} z_0}{n!} (z - z_0)^n \quad \text{—— (8)}$$

Clearly this series will converge and represent $f(z)$ if and only if

$$\lim_{n \to \infty} R_n(z) = 0$$

Since $|z - z_0| = r$, $|z' - z_0| = r_1$ and $|z' - z| \geq r_1 - r_2$ it follows from eqn (5) that, when M denotes the maximum value of $f(z')$ on C_1

$$|R_n| \leq \frac{r^{n+1}}{\cancel{2\pi}} \cdot \frac{M \cdot \cancel{2\pi} r_1}{r_1^{n+1} (r_1 - r)}$$

$$= \frac{r_1 M}{(r_1 - r)} \cdot \left(\frac{r}{r_1}\right)^{n+1} \quad \text{—— (9)}$$

But $\frac{r}{r_1} < 1$, therefore if we let n approach infinity, the expression on right approaches zero. This proves (9) for all values z inside C. It follows that series (8) converges and represents $f(z)$. It may be noted that the expansion (8) of $f(z)$ is based on the assumption that $f(z)$ is analytic for $|z - z_0| < r_0$. The representation of $f(z)$ in the form (8) is unique in the sense that (8) is the only power series with centre at z_0 which represents the given fun^n $f(z)$ and is the desired Taylor's series of $f(z)$ with centre at z_0.

This proves theorem derivatives of an analytic fun^n.

When $z_0 = 0$; the series (8) reduces to Maclaurin series

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^n(0)}{n!} z^n$$