B.sc Part I

Mathematics I (Hons)

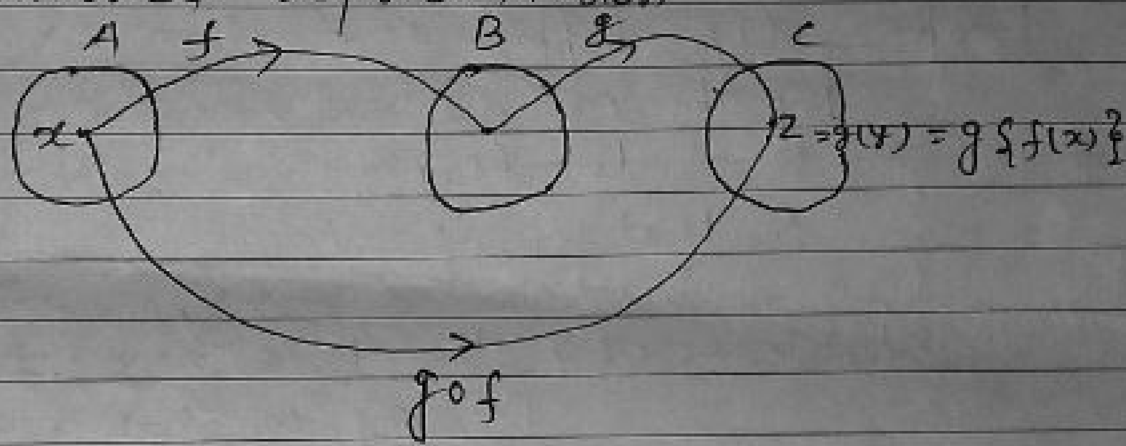
Composition

Definition: Let $f: A \to B$ and $g: B \to C$ be two functions. Then their composition $(g \circ f)$ is a function of $A \to C$ given by

$$(g \circ f)(x) = g\{f(x)\} \text{ for all } x \in A.$$

In other words, $g \circ f = \{(x, z) \mid$ there exists a y such that $(x, y) \in f$ and $(y, z) \in g\}$.

Diagrammatical representation



The functions f and g are collection of arcs joining points in A to points in B and the points in B to points in C.

The image of x in A is obtained by first finding the point $y = f(x)$ in B with the help of the function f and then finding the point $z = g(y) = g\{f(x)\}$ in C with the help of the function g.

Theorem: Let $f: X \to Y$, $g: Y \to Z$, $h: Z \to W$ be three functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

At the outset, it can be seen that both the compositions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are functions from $X \to W$.

Let $x \in X$

Now. $\{ho(gof)\}(x) = ho(gof(x)) = h(g(f(x))$

and $(hog)of(x) = (hog)(f(x)) = h(g(f(x)))$

thus $\{ho(gof)\}(x) = (hog)of(x)$.

Hence $ho(gof) = (hog)of$.

Theorem :- Let $f: X \to Y$ and $g: Y \to Z$. Then

(i) If each of f and g is one-one, gof is one-one

(ii) if each of f and g is onto, gof is onto.

Proof :- (i) Suppose that $x_1, x_2 \in X$ and that $x_1 \neq x_2$,

By the one-to-one nature of f, $f(x_1) \neq f(x_2)$

Let $f(x_1) = y_1$, $f(x_2) = y_2$

So that $y_1, y_2 \in Y$ and $y_1 \neq y_2$.

Since g is one to one and y_1, y_2 are distinct elements of Y, therefore $g(y_1) \neq g(y_2)$

i.e. $g(f(x_1)) \neq g(f(x_2))$

This shows that gof is one-one.

(ii) from the definition of surjection (onto),

Since f and g are onto mappings,

Therefore $f(X) = Y$, $g(Y) = Z$

Therefore $g\{f(X)\} = Z$, i.e. $\{gof\}(X) = Z$.

Hence gof is onto.

Proved