Group: – Let G be a non-empty set and o is binary operation then the system (G, o) is called group iff the following postulates are satisfied:

(i) closure law: – For all $a, b \in G$ such that $aob \in G$.

i.e. G is closed under the binary operation o.

(ii) Associative law: – For all $a, b, c \in G$ such that $(aob)oc = ao(boc)$

i.e. associative law for the operation o holds.

(iii) Existence of ~~identity~~ inverse elements: – For every $a \in G$ there exists an element $b \in G$ such that $aob = boa = e$, e being the identity element

i.e. the inverse elements of all the elements of G exist. The inverse of a denoted by a^{-1}.

(iv) Existence of identity element: – For all $a \in G$ and there exists an element $e \in G$ such that $aoe = eoa = a$.

i.e the identity element exist in G.

Theorem: – Show that the identity element in a group is unique.

or

Show that a group has a unique neutral element.

Proof: – Let suppose that a group (G, o) has two identity elements e and e', where $e \neq e'$.

When e is identity element then from the law of existence of identity element of a group, we have

$$aoe = eoa = a \quad \text{——— (1)} \quad \forall a \in G$$

And when e' is identity of G then

$$aoe' = e'oa = a \quad \text{——— (2)} \quad \forall a \in G$$

Here the equality of (1) and (2) show that $e = e'$

This lead our contradiction that $e \neq e'$. Hence our supposition is wrong. So the identity element of a group is unique.

2. Theorem :- Show that every element of a group has a unique inverse.

or

Prove that the inverse element of an element in a group is unique.

Proof :- Let (G, o) be a group and a be any element of G. Suppose x and y are two inverses of a then we have

$$a o x = x o a = e \quad [\because x \text{ is inverse of } a] \text{ --- } (1)$$

$$a o y = y o a = e \quad [\because y \text{ is inverse of } a] \text{ --- } (2)$$

from above two relations, we have

$$a o x = a o y = e$$

$$\Rightarrow x o (a o x) = x o (a o y) \quad [\text{multiply left by } x]$$

$$\Rightarrow [x o a] o x = (x o a) o y \quad (\text{By associative law})$$

$$\Rightarrow e o x = e o y \quad (\text{by } (1))$$

$$\Rightarrow x = y \quad [\because e \text{ is identity}]$$

This shows that any two inverse of 'a' must be identical. Hence every element of a group G is unique.